$$\int_C \frac{e^{\pi z}}{(2z-i)^3}\, dz = \frac{\pi i}{8} \cdot \pi^2 e^{\pi i/2} = \frac{\pi^3 i}{8}\,(\cos \pi/2 + i \sin \pi/2) = \frac{\pi^3 i}{8} \cdot i = \frac{-\pi^3}{8}$$

Thus $\displaystyle\int_C \frac{e^{\pi z}}{(2z-i)^3}\, dz = \frac{-\pi^3}{8}$

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32. *Evaluate* $\displaystyle\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\, dz$ *where* $C : |z| = 3$.

>> We shall first resolve $\dfrac{1}{(z+1)^2(z-2)}$ into partial fractions.

Let $\dfrac{1}{(z+1)^2(z-2)} = \dfrac{A}{z+1} + \dfrac{B}{(z+1)^2} + \dfrac{C}{z-2}$

or $1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$

Put $z = -1 : 1 = B(-3) \therefore B = -1/3$

Put $z = 2 : 1 = C(9) \therefore C = 1/9$

Put $z = 0 : 1 = A(-2) + B(-2) + C(1)$

$$1 = -2A + 2/3 + 1/9 \quad \therefore \quad A = -1/9$$

Now $\dfrac{1}{(z+1)^2(z-2)} = \dfrac{-1}{9} \cdot \dfrac{1}{z+1} - \dfrac{1}{3}\,\dfrac{1}{(z+1)^2} + \dfrac{1}{9} \cdot \dfrac{1}{z-2}$

Multiplying by e^{2z} and integrating w.r.t. z over C we have,

$$\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\, dz = \frac{-1}{9} \int_C \frac{e^{2z}}{z+1}\, dz - \frac{1}{3} \int_C \frac{e^{2z}}{(z+1)^2}\, dz + \frac{1}{9} \int_C \frac{e^{2z}}{z-2}\, dz \qquad \ldots(1)$$

The points $z = a = -1;\ z = a = 2$ lies inside the circle $|z| = 3$.

We shall consider Cauchy's integral formula in the forms

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a) \quad \text{and} \quad \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Taking $f(z) = e^{2z}$ we obtain $f'(z) = 2e^{2z}$

Now $\displaystyle\int_C \frac{e^{2z}}{z+1}\, dz = \int_C \frac{e^{2z}}{z-(-1)}\, dz = 2\pi i f(-1) = 2\pi i e^{-2} = \frac{2\pi i}{e^2}$

$$\int_C \frac{e^{2z}}{(z+1)^2}\,dz = \int_C \frac{e^{2z}}{[z-(-1)]^2}\,dz = \frac{2\pi i}{1!} f'(-1) = 2\pi i\,(2e^{-2})$$

ie., $$\int_C \frac{e^{2z}}{(z+1)^2}\,dz = \frac{4\pi i}{e^2}$$

Also $$\int_C \frac{e^{2z}}{z-2}\,dz = 2\pi i f(2) = 2\pi i \cdot e^4$$

Substituting these results in the RHS of (1) we obtain

$$\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz = \frac{-1}{9}\cdot\frac{2\pi i}{e^2} - \frac{1}{3}\cdot\frac{4\pi i}{e^2} + \frac{1}{9}\,2\pi i e^4$$

$$= \frac{-7}{9}\cdot\frac{2\pi i}{e^2} + \frac{2\pi i}{9}\,e^4$$

Thus $$\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz = \frac{2\pi i}{9}\left(e^4 - \frac{7}{e^2}\right)$$

---

33. *Evaluate* $\int_C \frac{dz}{(z^2+4)^2}$ *where* $C : |z-i| = 2$, *by Cauchy's integral formula.*

>> $C : |z-i| = 2$ is a circle with centre $(0, 1)$ and radius 2.

We have $$\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$$

Let $A = (0, 1)$ be the centre and $r = 2$ be the radius of C.

If $P_1 = (0, -2)$ and $P_2 = (0, 2)$ then $AP_1 = 3 > 2$ and $AP_2 = 1 < 2$

Hence $(0, 2)$ or $z = 2i$ only lies inside C.

We have Cauchy's integral formula in the form

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz \qquad \dots(1)$$

Now $$\frac{1}{(z^2+4)^2} = \frac{1}{[(z+2i)(z-2i)]^2} = \frac{1/(z+2i)^2}{(z-2i)^2}$$

Taking $f(z) = \dfrac{1}{(z+2i)^2}$ and $a = 2i$ we have

$$f'(z) = \frac{-2}{(z+2i)^3} \; ; \; f'(a) = f'(2i) = \frac{-2}{(4i)^3} = \frac{1}{32i}$$

Hence (1) becomes

$$\frac{1}{32i} = \frac{1}{2\pi i} \int_C \frac{1/(z+2i)^2}{(z-2i)^2} dz$$

ie., $$\frac{\pi}{16} = \int_C \frac{dz}{(z+2i)^2(z-2i)^2}$$

Thus $$\int_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}$$

---

34. *Evalute* $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$ *where C is the circle,*

(i) $|z| = 3$ *(ii)* $|z| = 1/2$ *(iii)* $|z| = 3/2$

>> We shall first resolve $\dfrac{1}{(z-1)^2(z-2)}$ into partial fractions.

Let $$\frac{1}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2} \qquad \ldots(1)$$

or $1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$

Put $z = 1 : \quad 1 = B(-1) \quad \therefore \quad B = -1$

Put $z = 2 : \quad 1 = C(1) \quad \therefore \quad C = 1$

Equating the coefficient of z^2 on both sides we have,

$$0 = A + C \quad \text{or} \quad A = -C \quad \therefore \; A = -1$$

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$

Multiplying (1) by $f(z)$ and intergrating w.r.t z over C by using the value of the constants obtained we have,

$$I = \int_C \frac{f(z)}{(z-1)^2(z-2)} dz = -\int_C \frac{f(z)}{z-1} dz - \int_C \frac{f(z)}{(z-1)^2} dz + \int_C \frac{f(z)}{z-2} dz$$

That is $I = I_1 + I_2 + I_3$ (*say*) $\qquad \ldots(2)$

Case-(i) $C : |z| = 3$

The points $z = 1$ and $z = 2$ both lie within C.

Hence by Cauchy's integral formula,

$$I_1 = -[2\pi i f(1)] = -2\pi i(\sin\pi + \cos\pi) = -2\pi i(0-1) = 2\pi i$$

$$I_2 = -[2\pi i f'(1)] \quad \text{But } f'(z) = 2\pi z(\cos\pi z^2 - \sin\pi z^2)$$

Hence $I_2 = -[2\pi i \cdot 2\pi(\cos\pi - \sin\pi)] = 4\pi^2 i$

$$I_3 = 2\pi i f(2) = 2\pi i(\sin 4\pi + \cos 4\pi) = 2\pi i(0+1) = 2\pi i$$

Hence from (2), $I = 2\pi i + 4\pi^2 i + 2\pi i = 4\pi i + 4\pi^2 i$

Thus $I = 4\pi i(1+\pi)$, where $C : |z| = 3$

Case- **(ii)** $C : |z| = 1/2$

The points $z = 1$ and $z = 2$ both lie outside C and hence $I_1 = 0 = I_2 = I_3$

Thus $I = 0$, where $C : |z| = 1/2$

Case- **(iii)** $C : |z| = 3/2$

The point $z = 1$ lies inside C and $z = 2$ lies outside C.

Hence $I_1 = 2\pi i f(1) = 2\pi i$, $I_2 = 2\pi i f'(1) = 4\pi^2 i$ and $I_3 = 0$

Now, $I = 2\pi i + 4\pi^2 i + 0 = 2\pi i(1+2\pi)$

Thus $I = 2\pi i(1+2\pi)$, where $C : |z| = 3/2$

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35. *Evaluate* $\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\, dz$ *where C is the circle* $|z| = 1$.

>> We have $f^{(n)}(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}}\, dz$ . . . (1)

The point $z = a = \pi/6 \approx 0.5$ lies with in the circle $|z| = 1$.

Now by putting $n = 2$ in (1) we have,

$$f^{(2)}(a) = f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3}\, dz$$

Taking $f(z) = \sin^6 z$ we have with $a = \pi/6$

$$f''(\pi/6) = \frac{1}{\pi i} \int_C \frac{\sin^6 z}{(z-\pi/6)^3}\, dz \qquad \ldots (2)$$

Consider $f(z) = \sin^6 z$

$\therefore \quad f'(z) = 6\sin^5 z\cos z \; ; \; f''(z) = -6\sin^6 z + 30\sin^4 z\cos^2 z$

Now $f''(\pi/6) = -6\sin^6(\pi/6) + 30\sin^4(\pi/6)\cos^2(\pi/6)$

ie., $f''(\pi/6) = -6\cdot\left(\frac{1}{2}\right)^6 + 30\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{-6}{64} + \frac{90}{64} = \frac{84}{64} = \frac{21}{16}$

Thus by substituting this value in (2) we have $\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\, dz = \frac{21\pi i}{16}$

EXERCISES

1. Evaluate $\int_C (z^2+z)\, dz$ along the line joining $(1-i)$ and $(2+3i)$.

2. Evaluate $\int_C (\bar{z})^2\, dz$ where

   (i) C is the circle $|z-1| = 1$.

   (ii) C is the circle $|z-2| = 1$.

3. Evaluate $\int_0^{1+i} (x^2 - iy)\, dz$ along the paths (1) $y = x$ (2) $y = x^2$

4. Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ where C is the square having vertices as $1 \pm i,\ -1 \pm i$

5. Verify Cauchy's theorem for $f(z) = z^2$ over the square formed by the points $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.

6. Evaluate $\int_C \frac{dz}{4z^2-9}$ where C is the circle $|z| = 2$

7. Evaluate $\int_C \frac{z\, dz}{(z^2+1)(z^2-9)}$ where C is the circle

   (i) $|z| = 2$ (ii) $|z-2| = 2$

8. Evaluate $\int_C \frac{(z-1)\,dz}{(z+1)^2(z-2)}$ where C is the circle $|z-i| = 2$.

9. Evaluate $\int_C \frac{e^z\,dz}{(z^2+\pi^2)^2}$ where C is the circle $|z| = 4$.

10. Evaluate $\int_C \frac{z^2+1}{z^2-1}\,dz$ where C is the circle :

(i) $|z-1| = 1$ (ii) $|z+1| = 1$

ANSWERS

1. $-\frac{1}{6}(103-64i)$

2. (i) $4\pi i$ (ii) $8\pi i$

3. (i) $\frac{1}{6}(5-i)$ (ii) $\frac{1}{6}(5+i)$

6. 0

7. (i) $-\pi i/5$ (ii) $\pi i/10$

8. $-2\pi i/9$

9. i/π

10. $2\pi i,\ -2\pi i$